Algebra 2 Name Key

3-6: Factor and Rational Root Theorem NOTES

Objectives: Use the Rational Root Theorem to identify potential zeroes.

The Rational Root Theorem (RRT): The Rational Root Theorem gives us a tool to predict which values are potential solutions when none are given.

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ be a polynomial in standard form with integer coefficients. There are limited number of possible roots of P(x)=0.

- Integer roots must be factors of a_0
- Rational roots must have a reduced form where p is an integer factor of a_0 and q is an integer factor of a_n.

$\frac{p}{q} = possible\ roots = \frac{factors\ of\ constant\ term}{factors\ of\ leading\ coefficent}$

$\frac{\text{all last}}{\text{all first}}$

That is: Divide all factors of the last number by all factors of the lead number

Example 1 List the Possible Rational Roots of $P(x) = 8x^5 - 32x^4 + x^2 - 4$

Last: 4 $\pm 1, \pm 2, \pm 4$ ← all Factors

First: 8 $\pm 1, \pm 2, \pm 4, 8$

$\frac{\text{all Last}}{\text{all First}}$: $\pm \frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{2}{1}, \frac{2}{2}, \frac{2}{4}, \frac{2}{8}, \frac{4}{1}, \frac{4}{2}, \frac{4}{4}, \frac{4}{8} \rightarrow \boxed{\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}, \pm 2, \pm 4}$

You try. List the Possible Rational Roots of:

a. $4x^4 + 13x^3 - 124x^2 + 212x - 8 = 0$

Last 8 $\pm 1, 2, 4, 8$

First 4 $\pm 1, 2, 4$

$\frac{L}{F}$ $\frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{2}{1}, \frac{2}{2}, \frac{2}{4}, \frac{4}{1}, \frac{4}{2}, \frac{4}{4}, \frac{8}{1}, \frac{8}{2}, \frac{8}{4} \rightarrow \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 2, \pm 4, \pm 8$

b. $7x^4 + 13x^3 - 124x^2 + 212x - 45 = 0$

L: 45 1, 3, 5, 9, 15, 45

F: 7 1, 7

$\frac{L}{F}$: $\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45$

$\pm \frac{1}{7}, \pm \frac{3}{7}, \pm \frac{5}{7}, \pm \frac{9}{7}, \pm \frac{15}{7}$

$\pm 45/7$

Example 2 A storage company is designing a new storage unit. Based on the dimensions shown, the volume of a container is modeled by the polynomial $v(x) = 2x^3 - 7x^2 + 6x$ where x is the width in feet. What are the dimensions of the container in feet if the volume of the unit is 154 ft³?

$154 = 2x^3 - 7x^2 + 6x$

$0 = 2x^3 - 7x^2 + 6x - 154$

graph on Desmos $x = 5.5 \rightarrow 11/2$

L 154: 1, 2, 7, 11, 14, 22, 77, 154

F 2 1, 2

$\frac{L}{F}$: $\pm 1, 2, 7, 11, 14, 22, 77, 154$

$\pm 1/2, 7/2, 11/2, 77/2$

Fundamental Theorem of algebra

if Polynomial, n degree, then it has n solutions, some may be repeats, or imaginary

so x^4 has 4 solutions.

Example 3. Put it all together. Use the RRT, synthetic division, and factoring to find all the rational solutions of $3x^4+4x^3+2x^2-x-2=0$. $x^4 \rightarrow$ so 4 answers

List potential factors:

L: 2 $\pm 1, 2$ $\frac{L}{F}$ $\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

F: 3 $\pm 1, 3$

Use synthetic division to find one with remainder of 0. $x=-1$ works $x=\frac{2}{3}$

Solve the depressed equation with synthetic division, factoring or quad formula.

$$\begin{array}{r|rrrrr} -1 & 3 & 4 & 2 & -1 & -2 \\ & \downarrow & -3 & -1 & -1 & 2 \\ \hline & 3 & 1 & 1 & -2 & 0 \end{array}$$

x^3

$$\begin{array}{r|rrrr} \frac{2}{3} & 3 & 1 & 1 & -2 \\ & & 2 & 2 & 2 \\ \hline & 3 & 3 & 3 & 0 \end{array}$$

x^2

$3x^2+3x+3=0$

$x=\frac{-3\pm\sqrt{9-4(3)(3)}}{6}$

$x=\frac{-3\pm\sqrt{-27}}{6}=\frac{-3\pm 3\sqrt{3}i}{6}=\boxed{\frac{-1\pm\sqrt{3}i}{2}}$

You Try:

Use the RRT, synthetic division, and factoring to find all rational solutions of $x^3-2x^2+5x-10=0$

L: 10 $\pm 1, \pm 2, \pm 5, \pm 10$

F: 1

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 5 & -10 \\ & \downarrow & 2 & 0 & 10 \\ \hline & 1 & 0 & 5 & 0 \end{array}$$

$x^2+5=0$

$x^2=-5$

$\boxed{x=\pm i\sqrt{5}}$

$x^2(x-2) + 5(x-2)=0$

$(x^2+5)(x-2)=0$

$x^2+5=0$ $x-2=0$

$x^2=-5$ $x=2$

$x=\pm\sqrt{5}i$

When solving a polynomial, look to factor first. If not factorable, then use the RRT and synthetic division until you can either factor or solve another way.

Conjugate Roots Theorem

if $a+bi$ is a root then $a-bi$ is a root
(imaginary must be in pairs).
if $a+\sqrt{b}$ is a root, $a-\sqrt{b}$ is a root.
$\sqrt{\ }$square roots must be in pairs.

Example 5. Write a polynomial using conjugates and zeroes. see back for A) B)

A) What is a quadratic function ***P*** with rational coefficients in standard form such that P(x)=0 has 2+5i as a root? if $2+5i$ is a root $2-5i$ is a root

$x=2+5i$ $\quad$ $x=2-5i$

$x-2-5i=0$ $\quad$ $x-2+5i=0$

Factor $\quad$ Factor

$(x-2-5i)(x-2+5i)=0$ multiply

$x^2-2x+5ix+4-10i$
$-2x-5ix \quad +10i-25i^2$

$x^2-4x+4-25i^2$ $\quad$ $i^2=-1$

$x^2-4x+4+25=0$

$\boxed{x^2-4x+29=0}$

B) A polynomial function ***Q*** of degree 4 with rational coefficients has zeros $3-\sqrt{7}$ and 4i. What is a polynomial equation in standard form with these roots?

$x=4i$ $\quad$ $x=-4i$

$x-4i=0$ $\quad$ $x+4i=0$

$(x-4i)(x+4i)$

$x^2-4ix+4ix-16i^2$

x^2-16i^2

(x^2+16)

$x=3-\sqrt{7}$ $\quad$ $x=3+\sqrt{7}$

$x-3+\sqrt{7}=0$ $\quad$ $x-3-\sqrt{7}=0$

$(x-3+\sqrt{7})(x-3-\sqrt{7})$

$x^2-3x-\sqrt{7}x+9+3\sqrt{7}$
$-3x+\sqrt{7}x \quad -3\sqrt{7}-\sqrt{49}$

$x^2-6x+9-\sqrt{49}$

$x^2-6x+9-7=0$

$x^2-6x+2=0$

$(x^2+16)(x^2-6x+2)=0$

$x^4-6x^3+2x^2$
$+16x^2-96x+32$

$\boxed{x^4-6x^3+18x^2-96x+32=0}$

A) $x = 2i \quad x = -2i$

$(x-2i)(x+2i) = 0$

$x^2 - 2ix + 2ix - 4i^2 = 0$

$x^2 - 4i^2 = 0$

$x^2 + 4 = 0$

B) $x = 1 - \sqrt{3} \quad x = 1 + \sqrt{3}$

$(x - 1 + \sqrt{3}) = 0 \quad x - 1 - \sqrt{3} = 0$

$(x - 1 + \sqrt{3})(x - 1 - \sqrt{3}) = 0$

$x^2 - x - \sqrt{3}x + 1 + \sqrt{3}$

$\quad - x + \sqrt{3}x \quad - \sqrt{3} - \sqrt{9}$

$x^2 - 2x + 1 - \sqrt{9} = 0$

$x^2 - 2x + 1 - 3 = 0$

$\boxed{x^2 - 2x - 2 = 0}$